

# Maximally Monotone Linear Subspace Extensions of Monotone Subspaces: Explicit Constructions and Characterizations

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Dedicated to Jonathan Borwein on the occasion of his 60th birthday.

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## Abstract

Monotone linear relations play important roles in variational inequality problems and quadratic optimizations. In this paper, we give explicit maximally monotone linear subspace extensions of a monotone linear relation in finite dimensional spaces. Examples are provided to illustrate our extensions. Our results generalize a recent result by Crouzeix and Anaya.

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## 1 Introduction

Throughout this paper, we assume that  $\mathbb{R}^n$  ( $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ ) is an Euclidean space with the inner product  $\langle \cdot, \cdot \rangle$ , and induced Euclidean norm  $\| \cdot \|$ . Let  $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a

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*set-valued operator* from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , i.e., for every  $x \in \mathbb{R}^n$ ,  $Gx \subseteq \mathbb{R}^n$ , and let  $\text{gra } G = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid x^* \in Gx\}$  be the *graph* of  $G$ . Recall that  $G$  is *monotone* if

$$(1) \quad (\forall (x, x^*) \in \text{gra } G) (\forall (y, y^*) \in \text{gra } G) \quad \langle x - y, x^* - y^* \rangle \geq 0,$$

and *maximally monotone* if  $G$  is monotone and  $G$  has no proper monotone extension (in the sense of graph inclusion). We say that  $G$  is a *linear relation* if  $\text{gra } G$  is a linear subspace. While linear relations have been extensively studied [9, 18, 6, 2, 3, 17], monotone operators are ubiquitous in convex optimization and variational analysis [1, 15, 6, 7].

The central object of this paper is to consider the linear relation  $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ :

$$(i) \quad \text{gra } G = \{(x, x^*) \mid Ax + Bx^* = 0\}.$$

$$(ii) \quad A, B \in \mathbb{R}^{p \times n}.$$

$$(iii) \quad \text{rank}(A \ B) = p.$$

Our main concern is to find explicit maximally monotone linear subspace extensions of  $G$ . Recently, finding constructive maximal monotone extensions instead of using Zorn's lemma has been a very active topic [5, 4, 12, 11, 10]. In [10], Crouzeix and Anaya gave an algorithm to find maximally monotone linear subspace extensions of  $G$ , but it is not clear what the maximally monotone extensions are analytically. In this paper, we provide some maximally monotone extensions of  $G$  with closed analytical forms. Along the way, we also give a new proof to Crouzeix and Anaya's characterizations on monotonicity and maximal monotonicity of  $G$ . Our key tool is the Brezis-Browder characterization of maximally monotone linear relations.

The paper is organized as follows. In the remainder of this introductory section, we describe some central notions fundamental to our analysis. In Section 2, we collect some auxiliary results for future reference and for the reader's convenience. Section 3 provides explicit self-dual maximal monotone extensions by using subspaces on which  $AB^\top + BA^\top$  is negative semidefinite, and obtain a complete characterization of all maximal monotone extensions. Section 4 deals with Minty's parameterizations of monotone operator  $G$ . In Section 5, we get some explicit maximally monotone extensions with the same domain or the same range by utilizing normal cone operators. In Section 6, we illustrate our maximally monotone extensions by considering three examples.

Our notations are standard. We use  $\text{dom } G = \{x \in \mathbb{R}^n \mid Gx \neq \emptyset\}$  for the *domain* of  $G$ ,  $\text{ran } G = G(\mathbb{R}^n)$  for the *range* of  $G$  and  $\text{ker } G = \{x \in \mathbb{R}^n \mid 0 \in Gx\}$  for the *kernal* of  $G$ . Given a subset  $C$  of  $\mathbb{R}^n$ ,  $\text{span } C$  is the *span* (the set of all finite linear combinations) of  $C$ . We set

$$C^\perp = \{x^* \in \mathbb{R}^n \mid (\forall c \in C) \langle x^*, c \rangle = 0\}.$$

Then the *adjoint* of  $G$ , denoted by  $G^*$ , is defined by

$$\text{gra } G^* = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid (x^*, -x) \in (\text{gra } G)^\perp\}.$$

The set  $\mathbb{R}^{n \times p}$  is the set of all the  $n \times p$  matrices, for  $n, p \in \mathbb{N}$ . Then  $\text{rank}(M)$  is the *rank* of the matrix  $M \in \mathbb{R}^{n \times p}$ . Let  $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the identity mapping, i.e.,  $\text{Id } x = x$  for  $x \in \mathbb{R}^n$ . We also set  $P_X : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : (x, x^*) \mapsto x$ , and  $P_{X^*} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : (x, x^*) \mapsto x^*$ . If  $\mathcal{X}, \mathcal{Y}$  are subspaces of  $\mathbb{R}^n$ , we let

$$\mathcal{X} + \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

Counting multiplicities, let

$$(2) \quad \lambda_1, \lambda_2, \dots, \lambda_k \text{ be all positive eigenvalues of } (AB^\top + BA^\top) \text{ and}$$

$$(3) \quad \lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_p \text{ be nonpositive eigenvalues of } (AB^\top + BA^\top).$$

Moreover, let  $v_i$  be an eigenvector of eigenvalue  $\lambda_i$  of  $(AB^\top + BA^\top)$  satisfying  $\|v_i\| = 1$ , and  $\langle v_i, v_j \rangle = 0$  for  $1 \leq i \neq j \leq q$ . It will be convenient to put

$$(4) \quad \text{Id}_\lambda = \text{diag}(\lambda_1, \dots, \lambda_p) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_p \end{pmatrix}, \quad V = [v_1 \ v_2 \ \cdots \ v_p].$$

## 2 Auxiliary results on linear relations

In this section, we collect some facts and preliminary results which will be used in sequel.

We first provide a result about subspaces on which a linear operator from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ , i.e, an  $n \times n$  matrix, is monotone. For  $M \in \mathbb{R}^{n \times n}$ , define three subspaces of  $\mathbb{R}^n$ , namely, the positive eigenspace, null eigenspace and negative eigenspace associated with  $M + M^\top$  by

$$\mathbf{V}_+(M) = \text{span} \left\{ w_1, \dots, w_s : \begin{array}{l} w_i \text{ is an eigenvector of positive eigenvalue } \alpha_i \text{ of } M + M^\top \\ \langle w_i, w_j \rangle = 0 \ \forall \ i \neq j, \|w_i\| = 1, i, j = 1, \dots, s. \end{array} \right\}$$

$$\mathbf{V}_0(M) = \text{span} \left\{ w_{s+1}, \dots, w_l : \begin{array}{l} w_i \text{ is an eigenvector of 0 eigenvalue of } M + M^\top \\ \langle w_i, w_j \rangle = 0, \ \forall \ i \neq j, \|w_i\| = 1, i, j = s+1, \dots, l. \end{array} \right\}$$

$$\mathbf{V}_-(M) = \text{span} \left\{ w_{l+1}, \dots, w_n : \begin{array}{l} w_i \text{ is an eigenvector of negative eigenvalue } \alpha_i \text{ of } M + M^\top \\ \langle w_i, w_j \rangle = 0 \ \forall i \neq j, \|w_i\| = 1, i, j = l+1, \dots, n. \end{array} \right\}$$

which is possible since a symmetric matrix always has a complete orthonormal set of eigenvectors, [14, pages 547–549].

**Proposition 2.1** *Let  $M$  be an  $n \times n$  matrix. Then*

- (i)  *$M$  is strictly monotone on  $\mathbf{V}_+(M)$ . Moreover,  $M + M^\top : \mathbf{V}_+(M) \rightarrow \mathbf{V}_+(M)$  is a bijection.*
- (ii)  *$M$  is monotone on  $\mathbf{V}_+(M) + \mathbf{V}_0(M)$ .*
- (iii)  *$-M$  is strictly monotone on  $\mathbf{V}_-(M)$ . Moreover,  $-(M + M^\top) : \mathbf{V}_-(M) \rightarrow \mathbf{V}_-(M)$  is a bijection.*
- (iv)  *$-M$  is monotone on  $\mathbf{V}_-(M) + \mathbf{V}_0(M)$ .*
- (v) *For every  $x \in \mathbf{V}_0(M)$ ,  $(M + M^\top)x = 0$  and  $\langle x, Mx \rangle = 0$ .*

*In particular, the orthogonal decomposition holds:  $\mathbb{R}^n = \mathbf{V}_+(M) \oplus \mathbf{V}_0(M) \oplus \mathbf{V}_-(M)$ .*

*Proof.* (i): Let  $x \in \mathbf{V}_+(M)$ . Then  $x = \sum_{i=1}^s l_i w_i$  for some  $(l_1, \dots, l_s) \in \mathbb{R}^s$ . Since  $\{w_1, \dots, w_s\}$  is a set of orthonormal vectors, they are linearly independent so that

$$x \neq 0 \iff (l_1, \dots, l_s) \neq 0.$$

Note that  $\alpha_i > 0$  when  $i = 1, \dots, s$  and  $\langle w_i, w_j \rangle = 0$  for  $i \neq j$ . We have

$$\begin{aligned} 2\langle x, Mx \rangle &= \langle x, (M + M^\top)x \rangle = \left\langle \sum_{i=1}^s l_i w_i, (M + M^\top) \left( \sum_{i=1}^s l_i w_i \right) \right\rangle \\ &= \left\langle \sum_{i=1}^s l_i w_i, \sum_{i=1}^s l_i \alpha_i w_i \right\rangle = \sum_{i=1}^s \alpha_i l_i^2 > 0 \end{aligned}$$

if  $x \neq 0$ .

For every  $x \in \mathbf{V}_+(M)$  with  $x = \sum_{i=1}^s l_i w_i$ , we have

$$(M + M^\top)x = \sum_{i=1}^s l_i (M + M^\top)w_i = \sum_{i=1}^s \alpha_i l_i w_i \in \mathbf{V}_+(M).$$

As  $\alpha_i > 0$  for  $i = 1, \dots, s$  and  $\{w_1, \dots, w_s\}$  is an orthonormal basis of  $\mathbf{V}_+(M)$ , we conclude that  $M + M^\top : \mathbf{V}_+(M) \rightarrow \mathbf{V}_+(M)$  is a bijection.

(ii): Let  $x \in \mathbf{V}_+(M) + \mathbf{V}_0(M)$ . Then  $x = \sum_{i=1}^l l_i w_i$  for some  $(l_1, \dots, l_l) \in \mathbb{R}^l$ . Note that  $\alpha_i \geq 0$  when  $i = 1, \dots, l$  and  $\langle w_i, w_j \rangle = 0$  for  $i \neq j$ . We have

$$\begin{aligned} 2\langle x, Mx \rangle &= \langle x, (M + M^\top)x \rangle = \left\langle \sum_{i=1}^l l_i w_i, (M + M^\top) \left( \sum_{i=1}^l l_i w_i \right) \right\rangle \\ &= \left\langle \sum_{i=1}^l l_i w_i, \sum_{i=1}^l l_i \alpha_i w_i \right\rangle = \sum_{i=1}^l \alpha_i l_i^2 \geq 0. \end{aligned}$$

The proofs for (iii), (iv) are similar as (i), (ii).

(v): For  $x \in \mathbf{V}_0(M)$ ,

$$2\langle x, Mx \rangle = \langle x, (M + M^\top)x \rangle = \langle x, 0 \rangle = 0.$$

■

**Corollary 2.2** *Then following holds.*

(i)

$$\text{gra } T = \{(B^\top u, A^\top u) \mid u \in \mathbf{V}_+(BA^\top)\}$$

*is strictly monotone.*

(ii)

$$\text{gra } T = \{(B^\top u, A^\top u) \mid u \in \mathbf{V}_+(BA^\top) + \mathbf{V}_0(BA^\top)\}$$

*is monotone.*

(iii)

$$\text{gra } T = \{(B^\top u, -A^\top u) \mid u \in \mathbf{V}_-(BA^\top)\}$$

*is strictly monotone.*

(iv)

$$\text{gra } T = \{(B^\top u, -A^\top u) \mid u \in \mathbf{V}_-(BA^\top) + \mathbf{V}_0(BA^\top)\}$$

*is monotone.*

*Proof.* It follows from Proposition 2.1 and  $\langle B^\top u, A^\top u \rangle = \langle u, BA^\top u \rangle$ ,  $\forall u \in \mathbb{R}^n$ . ■

**Lemma 2.3** *For every subspace  $S \subseteq \mathbb{R}^p$ , the following hold.*

(5)

$$\dim\{(B^\top u, A^\top u) \mid u \in S\} = \dim S.$$

(6)

$$\dim\{(B^\top u, -A^\top u) \mid u \in S\} = \dim S.$$

*Proof.* Let  $\dim S = t$  and  $\{u_1, \dots, u_t\}$  be a basis of  $S$ . We claim that the set of vectors

$$\left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u_i \mid i = 1, \dots, t \right\}$$

is linearly independent. Indeed, because  $(A \ B)$  has full row rank  $p$ ,

$$\sum_{i=1}^t l_i \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u_i = \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} \sum_{i=1}^t l_i u_i = 0 \quad \Leftrightarrow \quad \sum_{i=1}^t l_i u_i = 0 \quad \Leftrightarrow \quad l_i = 0 \text{ for } i = 1, \dots, t.$$

Note that

$$\begin{pmatrix} B^\top u \\ -A^\top u \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \begin{pmatrix} B^\top u \\ A^\top u \end{pmatrix}$$

and

$$\begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}$$

is invertible, we have

$$\dim\{(B^\top u, A^\top u) \mid u \in S\} = \dim\{(B^\top u, -A^\top u) \mid u \in S\}$$

so (6) follows from (5). Alternatively, see [14, page 208, Exercise 4.49]. ■

**Fact 2.4** *We have*

$$(AB^\top + BA^\top)V = V \text{Id}_\lambda.$$

*Proof.* Let  $y = [y_1 \ y_2 \ \dots \ y_p]^\top \in \mathbb{R}^p$ . Then we have

$$(AB^\top + BA^\top)Vy = (AB^\top + BA^\top)\left(\sum_{i=1}^p y_i v_i\right) = \sum_{i=1}^p \lambda_i y_i v_i = V \text{Id}_\lambda y.$$

■

Two key criteria concerning maximally monotone linear relations come as follows:

**Fact 2.5** (See [19, Proposition 4.2.9 ] or [3, Proposition 2.10].) *Let  $T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a monotone linear relation. The following are equivalent:*

- (i)  $T$  is maximally monotone.
- (ii)  $\dim \text{gra } T = n$ .
- (iii)  $\text{dom } T = (T0)^\perp$ .

**Fact 2.6 (Brézis-Browder)** (See [8, Theorem 2], or [18] or [16].) *Let  $T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a monotone linear relation. Then the following statements are equivalent.*

- (i)  $T$  is maximally monotone.
- (ii)  $T^*$  is maximal monotone.
- (iii)  $T^*$  is monotone.

Some basic properties of  $G$  are:

**Lemma 2.7** (i)  $\text{gra } G = \ker(A \ B)$ .

(ii)  $G0 = \ker B$ ,  $G^{-1}(0) = \ker A$ .

(iii)  $\text{dom } G = P_X(\ker(A \ B))$  and  $\text{ran } G = P_{X^*}(\ker(A \ B))$ .

(iv)  $\text{ran}(G + \text{Id}) = P_{X^*}(\ker(A - B \ B)) = P_X(\ker(A \ B - A))$ , and

$$\text{dom } G = P_X(\ker(A - B \ B)), \quad \text{ran } G = P_{X^*}(\ker(A \ (B - A))).$$

(v)  $\dim G = 2n - p$ .

*Proof.* (i), (ii), (iii) follow from definition of  $G$ . Since

$$Ax + Bx^* = 0 \quad \Leftrightarrow \quad (A - B)x + B(x + x^*) = 0 \quad \Leftrightarrow \quad A(x + x^*) + (B - A)x^* = 0,$$

(iv) holds.

(v): We have

$$2n = \dim \ker(A \ B) + \dim \text{ran} \begin{pmatrix} A^\top \\ B^\top \end{pmatrix} = \dim G + p.$$

Hence  $\dim G = 2n - p$ . ■

The following result summarizes the monotonicities of  $G^*$  and  $G$ .

**Lemma 2.8** *The following hold.*

- (i)  $\text{gra } G^* = \{(B^\top u, -A^\top u) \mid u \in \mathbb{R}^p\}$ .
- (ii)  $G^*$  is monotone  $\Leftrightarrow A^\top B + B^\top A$  is negative-semidefinite.

(iii) Assume  $G$  is monotone. Then  $n \leq p$ . Moreover,  $G$  is maximally monotone if and only  $\dim G = n = p$ .

*Proof.* (i): By Lemma 2.7(i), we have

$$(x, x^*) \in \text{gra } G^* \Leftrightarrow (x^*, -x) \in \text{gra } G^\perp = \text{ran} \begin{pmatrix} A^\top \\ B^\top \end{pmatrix} = \{(A^\top u, B^\top u) \mid u \in \mathbb{R}^p\}.$$

Thus  $\text{gra } G^* = \{(B^\top u, -A^\top u) \mid u \in \mathbb{R}^p\}$ .

(ii): Since  $\text{gra } G^*$  is a linear subspace, by (i),

$$\begin{aligned} G^* \text{ is monotone} &\Leftrightarrow \langle B^\top u, -A^\top u \rangle \geq 0, \quad \forall u \in \mathbb{R}^p \Leftrightarrow \langle u, -BA^\top u \rangle \geq 0, \quad \forall u \in \mathbb{R}^p \\ &\Leftrightarrow \langle u, BA^\top u \rangle \leq 0, \quad \forall u \in \mathbb{R}^p \Leftrightarrow \langle u, (A^\top B + B^\top A)u \rangle \leq 0, \quad \forall u \in \mathbb{R}^p \\ &\Leftrightarrow (A^\top B + B^\top A) \text{ is negative semidefinite.} \end{aligned}$$

(iii): By Fact 2.5 and Lemma 2.7(v),  $2n - p = \dim \text{gra } G \leq n \Rightarrow n \leq p$ . By Fact 2.5 and Lemma 2.7(v) again,  $G$  is maximally monotone  $\Leftrightarrow 2n - p = \dim \text{gra } G = n \Leftrightarrow \dim \text{gra } G = p = n$ . ■

### 3 Explicit maximal monotone extensions of monotone linear relations

In this section, we give explicit maximal monotone linear subspace extensions of  $G$  by using  $\mathbf{V}_+(AB^\top)$  or  $V_g$ . A characterization of all the maximally monotone extensions of  $G$  is also given.

**Lemma 3.1** Let  $N \in \mathbb{R}^{p \times p}$  and  $\tilde{G}$  and  $\hat{G}$  be defined by

$$\begin{aligned} \text{gra } \tilde{G} &= \{(x, x^*) \mid N^\top V^\top A x + N^\top V^\top B x^* = 0\} \\ \text{gra } \hat{G} &= \{(B^\top u, -A^\top u) \mid u \in \text{ran } VN\}. \end{aligned}$$

Then  $(\tilde{G})^* = \hat{G}$ .

*Proof.* Let  $(y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$ . Then we have

$$\begin{aligned} (y, y^*) \in \text{gra}(\tilde{G})^* &\Leftrightarrow (y^*, -y) \in (\text{gra } \tilde{G})^\perp = (\ker \begin{pmatrix} N^\top V^\top A & N^\top V^\top B \end{pmatrix})^\perp = \text{ran} \begin{pmatrix} A^\top VN \\ B^\top VN \end{pmatrix} \\ &\Leftrightarrow (y, y^*) \in \text{gra } \hat{G}. \end{aligned}$$

Hence  $(\tilde{G})^* = \hat{G}$ . ■



**Lemma 3.2** Define  $\tilde{G}$  and  $\hat{G}$  by

$$\begin{aligned}\text{gra } \tilde{G} &= \{(x, x^*) \mid V_g Ax + V_g Bx^* = 0\} \\ \text{gra } \hat{G} &= \{(B^\top u, -A^\top u) \mid u \in \mathbf{V}_-(BA^\top) + \mathbf{V}_0(BA^\top)\},\end{aligned}$$

where  $V_g$  is  $(p-k) \times p$  matrix defined by

$$V_g = \begin{pmatrix} v_{k+1}^\top \\ v_{k+2}^\top \\ \vdots \\ v_p^\top \end{pmatrix}.$$

Then

- (i)  $\hat{G}$  is monotone.
- (ii)  $(\hat{G})^* = \tilde{G}$ .
- (iii)  $\text{gra } \tilde{G} = \text{gra } G + \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in \mathbf{V}_+(BA^\top) \right\}.$

*Proof.* (i): Apply Corollary 2.2(iv).

(ii): Notations are as in (4). Let

$$(7) \quad N = [0 \ 0 \cdots 0 \ e_{k+1} \cdots e_p],$$

where  $e_i = [0, 0 \cdots 1, 0 \cdots 0]^\top$ : the  $i$ th entry is 1 and the others are 0.

Then we have

$$(8) \quad N^\top V^\top = \left( (v_1 \ \cdots v_k \ V_g^\top) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Id} \end{pmatrix} \right)^\top = \begin{pmatrix} 0 \\ V_g \end{pmatrix}.$$

Then we have

$$\begin{aligned}V_g Ax + V_g Bx^* = 0 &\Leftrightarrow \begin{pmatrix} 0 \\ V_g Ax + V_g Bx^* \end{pmatrix} = 0 \\ &\Leftrightarrow N^\top V^\top Ax + N^\top V^\top Bx^* = 0, \quad \forall (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n.\end{aligned}$$

Hence

$$\text{gra } \tilde{G} = \{(x, x^*) \mid N^\top V^\top Ax + N^\top V^\top Bx^* = 0\}.$$

Thus by Lemma 3.1,

$$\text{gra}(\tilde{G})^* = \{(B^\top u, -A^\top u) \mid u \in \text{ran } VN = \text{ran} \begin{pmatrix} 0 & V_g^\top \end{pmatrix} = \mathbf{V}_-(BA^\top) + \mathbf{V}_0(BA^\top)\} = \text{gra } \widehat{G}.$$

Hence  $(\widehat{G})^* = (\tilde{G})^{**} = \tilde{G}$ .

(iii): Let  $J$  be defined by

$$\text{gra } J = \text{gra } G + \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in \mathbf{V}_+(BA^\top) \right\}.$$

Then we have

$$(\text{gra } J)^\perp = (\text{gra } G)^\perp \cap \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in \mathbf{V}_+(BA^\top) \right\}^\perp.$$

By Lemma 2.7(i),

$$\text{gra } G^\perp = \left\{ \begin{pmatrix} A^\top \\ B^\top \end{pmatrix} w \mid w \in \mathbb{R}^p \right\}$$

Then

$$\begin{pmatrix} A^\top \\ B^\top \end{pmatrix} w \in \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in \mathbf{V}_+(BA^\top) \right\}^\perp$$

if and only if

$$\langle (A^\top w, B^\top w), (B^\top u, A^\top u) \rangle = 0 \quad \forall u \in \mathbf{V}_+(BA^\top),$$

that is,

$$(9) \quad \langle A^\top w, B^\top u \rangle + \langle B^\top w, A^\top u \rangle = \langle w, (AB^\top + BA^\top)u \rangle = 0 \quad \forall u \in \mathbf{V}_+(AB^\top).$$

Because  $AB^\top + BA^\top : \mathbf{V}_+(AB^\top) \mapsto \mathbf{V}_+(AB^\top)$  is onto by Proposition 2.1(i), we obtain that (9) holds if and only if  $w \in \mathbf{V}_-(AB^\top) + \mathbf{V}_0(AB^\top)$ . Hence

$$(\text{gra } J)^\perp = \{(A^\top w, B^\top w) \mid w \in \mathbf{V}_-(BA^\top) + \mathbf{V}_0(BA^\top)\},$$

from which  $\text{gra } J^* = \text{gra } \widehat{G}$ . Then by (i),

$$\text{gra } \tilde{G} = \text{gra}(\widehat{G})^* = \text{gra } J^{**} = \text{gra } J.$$

■

We are ready to apply Brezis-Browder Theorem, namely Fact 2.6, to improve Crouzeix-Anaya's characterizations of monotonicity and maximal monotonicity of  $G$  and provide a different proof.

**Theorem 3.3** *Let  $\widehat{G}, \widetilde{G}$  be defined in Lemma 3.2. The following are equivalent:*

- (i)  $G$  is monotone;
- (ii)  $\widetilde{G}$  is monotone;
- (iii)  $\widetilde{G}$  is maximally monotone;
- (iv)  $\widehat{G}$  is maximally monotone;
- (v)  $\dim \mathbf{V}_+(BA^\top) = p - n$ , equivalently,  $AB^\top + BA^\top$  has exactly  $p - n$  positive eigenvalues (counting multiplicity).

*Proof.* (i) $\Leftrightarrow$ (ii): Lemma 3.2(iii) and Corollary 2.2(i).

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): Note that  $\widetilde{G} = (\widehat{G})^*$  and  $\widehat{G}$  is always a monotone linear relation by Corollary 2.2(iv). It suffices to combine Lemma 3.2 and Fact 2.6.

“(i) $\Rightarrow$ (v)” : Assume that  $G$  is monotone. Then  $\widetilde{G}$  is monotone by Lemma 3.2(iii) and Corollary 2.2(i). By Lemma 3.2(ii), Corollary 2.2(iv) and Fact 2.6,  $\widehat{G}$  is maximally monotone, so that  $\dim(\text{gra } \widehat{G}) = p - k = n$  by Fact 2.5 and Lemma 2.3, thus  $k = p - n$ . Note that for each eigenvalue of a symmetric matrix, its geometric multiplicity is the same as its algebraic multiplicity [14, page 512].

“(v) $\Rightarrow$ (i)” : Assume that  $k = p - n$ . Then  $\dim(\text{gra } \widehat{G}) = p - k = n$  by Lemma 2.3, so that  $\widehat{G}$  is maximally monotone by Fact 2.5(i)(ii). By Lemma 3.2(ii) and Fact 2.6,  $\widetilde{G}$  is monotone, which implies that  $G$  is monotone. ■

**Corollary 3.4** *Assume that  $G$  is monotone. Then*

$$\begin{aligned} \text{gra } \widetilde{G} &= \text{gra } G + \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in \mathbf{V}_+(BA^\top) \right\} \\ &= \{(x, x^*) \mid V_g Ax + V_g Bx^* = 0\} \end{aligned}$$

*is a maximally monotone extension of  $G$ , where*

$$V_g = \begin{pmatrix} v_{p-n+1}^\top \\ v_{p-n+2}^\top \\ \vdots \\ v_p^\top \end{pmatrix}.$$

*Proof.* Combine Theorem 3.3 and Lemma 3.2(iii) directly. ■

A remark is in order to compare our extension with the one by Crouzeix and Anaya.

**Remark 3.5** (i). Crouzeix-Anaya [10] defines the union of monotone extension of  $G$  as

$$S = \text{gra } G + \left\{ \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u \mid u \in K \right\}, \text{ where } K = \{u \in \mathbb{R}^n \mid \langle u, (AB^\top + BA^\top)u \rangle \geq 0\}.$$

Although this is the set monotonically related to  $G$ , it is not monotone in general as long as  $(AB^\top + BA^\top)$  has both positive eigenvalues and negative eigenvalues. Indeed, let  $(\alpha_1, u_1)$  and  $(\alpha_2, u_2)$  be eigen-pairs of  $(AB^\top + BA^\top)$  with  $\alpha_1 > 0$  and  $\alpha_2 < 0$ . We have

$$\langle u_1, (AB^\top + BA^\top)u_1 \rangle = \alpha_1 \|u_1\|^2 > 0, \quad \langle u_2, (AB^\top + BA^\top)u_2 \rangle = \alpha_2 \|u_2\|^2 < 0.$$

Choose  $\epsilon > 0$  sufficiently small so that

$$\langle u_1 + \epsilon u_2, (AB^\top + BA^\top)(u_1 + \epsilon u_2) \rangle > 0.$$

Then

$$\begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u_1, \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} (u_1 + \epsilon u_2) \in S.$$

However,

$$\begin{pmatrix} B^\top \\ A^\top \end{pmatrix} (u_1 + \epsilon u_2) - \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u_1 = \epsilon \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u_2$$

has

$$\langle \epsilon B^\top u_2, \epsilon A^\top u_2 \rangle = \epsilon^2 \langle u_2, BA^\top u_2 \rangle = \epsilon^2 \frac{\langle u_2, (AB^\top + BA^\top)u_2 \rangle}{2} < 0.$$

Therefore  $S$  is not monotone. By using  $\mathbf{V}_+(BA^\top) \subseteq K$ , we have obtained a maximally monotone extension of  $G$ .

(ii). Crouzeix and Anaya [10] find the maximal monotone linear subspace extension of  $G$  algorithmically by using  $u \in \widetilde{G}_k \setminus G_k$ . Computationally, it is not completely clear to us how to find such an  $u$ .

The following result extends the characterization of maximally monotone linear relations given by Crouzeix-Anaya [10].

**Theorem 3.6** *Let  $\widehat{G}, \widetilde{G}$  be defined in Lemma 3.2. The following are equivalent:*

- (i)  $G$  is maximally monotone;
- (ii)  $p = n$  and  $G$  is monotone;
- (iii)  $p = n$  and  $AB^\top + BA^\top$  is negative semidefinite.
- (iv)  $p = n$  and  $\widehat{G}$  is maximally monotone.

*Proof.* (i) $\Rightarrow$ (ii): Apply Lemma 2.8(iii).

(ii) $\Rightarrow$ (iii): Apply directly Theorem 3.3(i)(v).

(iii) $\Rightarrow$ (i): Assume that  $p = n$  and  $(AB^\top + BA^\top)$  is negative semidefinite. Then  $k = 0$  and  $\tilde{G} = G$ . It follows that  $\dim(\text{gra } \hat{G}) = p - k = n$  by Lemma 2.3, so that  $\hat{G}$  is maximally monotone by Corollary 2.2(iv) and Fact 2.5(i)(ii). Since  $(\hat{G})^* = \tilde{G}$  by Lemma 3.2(ii), Fact 2.6 gives that  $\tilde{G} = G$  is maximally monotone.

(iii) $\Rightarrow$ (iv): Assume that  $p = n$  and  $(AB^\top + BA^\top)$  is negative semidefinite. We have  $k = 0$  and  $\dim(\text{gra } \hat{G}) = p - k = n - 0 = n$ . Hence (iv) holds by Corollary 2.2(iv) and Fact 2.5(i)(ii).

(iv) $\Rightarrow$ (iii): Assume that  $\hat{G}$  is maximally monotone and  $p = n$ . We have  $\dim(\text{gra } \hat{G}) = p - k = n - k = n$  so that  $k = 0$ . Hence  $(AB^\top + BA^\top)$  is negative semidefinite.  $\blacksquare$

We end this section with a characterization of all the maximally monotone linear subspace extensions of  $G$ .

**Theorem 3.7** *Let  $G$  be monotone. Then  $\tilde{G}$  is a maximally monotone extension of  $G$  if and only if there exists  $N \in \mathbb{R}^{p \times p}$  with rank of  $n$  such that  $N^\top \text{Id}_\lambda N$  is negative semidefinite and*

$$(10) \quad \text{gra } \tilde{G} = \{(x, x^*) \mid N^\top V^\top Ax + N^\top V^\top Bx^* = 0\}.$$

*Proof.* “ $\Rightarrow$ ”: By Lemma 2.8(i), we have

$$(11) \quad \text{gra } G^* = \{(B^\top u, -A^\top u) \mid u \in \mathbb{R}^p\}.$$

Since  $\text{gra } G \subseteq \text{gra } \tilde{G}$  and thus  $\text{gra}(\tilde{G})^*$  is a subspace of  $\text{gra } G^*$ .

Thus by (11), there exists a subspace  $F$  of  $\mathbb{R}^p$  such that

$$(12) \quad \text{gra}(\tilde{G})^* = \{(B^\top u, -A^\top u) \mid u \in F\}.$$

By Fact 2.6, Fact 2.5 and Lemma 2.3, we have

$$(13) \quad \dim F = n.$$

Thus, there exists  $N \in \mathbb{R}^{p \times p}$  with rank  $n$  such that  $\text{ran } VN = F$  and

$$(14) \quad \text{gra}(\tilde{G})^* = \{(B^\top VN y, -A^\top VN y) \mid y \in \mathbb{R}^p\}.$$

As  $\tilde{G}$  is maximal monotone,  $(\tilde{G})^*$  is maximal monotone by Fact 2.6, so

$$N^\top V^\top (BA^\top + AB^\top) VN \text{ is negative semidefinite.}$$

Using Fact 2.4, we have

$$(15) \quad N^\top \text{Id}_\lambda N = N^\top V^\top V \text{Id}_\lambda N = N^\top V^\top (AB^\top + BA^\top) V N$$

which is negative semidefinite. (10) follows from (14) by Lemma 3.1.

“ $\Leftarrow$ ”: By Lemma 3.1, we have

$$(16) \quad \text{gra}(\tilde{G})^* = \{(B^\top V N u, -A^\top V N u) \mid u \in \mathbb{R}^p\}.$$

Observe that  $(\tilde{G})^*$  is monotone because  $N^\top V^\top (AB^\top + BA^\top) V N = N^\top \text{Id}_\lambda N$  is negative semidefinite by Fact 2.4 and the assumption. As  $\text{rank}(V N) = n$ , it follows from (16) and Lemma 2.3 that  $\dim \text{gra}(\tilde{G})^* = n$ . Therefore  $(\tilde{G})^*$  is maximally monotone by Fact 2.5. Applying Fact 2.6 for  $T = (\tilde{G})^*$  yields that  $\tilde{G} = (\tilde{G})^{**}$  is maximally monotone.  $\blacksquare$

From the above proof, we see that to find a maximal monotone extension of  $G$  one essentially need to find subspace  $F \subseteq \mathbb{R}^p$  such that  $\dim F = n$  and  $AB^\top + BA^\top$  is negative semidefinite on  $F$ . If  $F = \text{ran } M$  and  $M \in \mathbb{R}^{p \times p}$  with  $\text{rank } M = n$ , one can let  $N = V^\top M$ . The maximal monotone linear subspace extension of  $G$  is

$$\tilde{G} = \{(x, x^*) \mid M^\top A x + M^\top B x^* = 0\}.$$

In Corollary 3.4, one can choose  $M = (\underbrace{0 \ 0 \ \cdots 0}_n \quad v_{p-n+1} \ \cdots \ v_p)$ .

**Corollary 3.8** *Let  $G$  be monotone. Then  $\tilde{G}$  is a maximally monotone extension of  $G$  if and only if there exists  $M \in \mathbb{R}^{p \times p}$  with rank of  $n$  such that  $M^\top (AB^\top + BA^\top) M$  is negative semidefinite and*

$$(17) \quad \text{gra } \tilde{G} = \{(x, x^*) \mid M^\top A x + M^\top B x^* = 0\}.$$

Note that  $G$  may have different representations in terms of  $A, B$ . The maximal monotone extension of  $\tilde{G}$  given in Theorem 3.7 and Corollary 3.4 relies on  $A, B$  matrices and  $N$ . This might leads different maximal monotone extensions, see Section 6.

## 4 Minty parameterizations

Although  $G$  is set-valued in general, when  $G$  is monotone it has a beautiful Minty parametrization in terms of  $A, B$ , which is what we are going to show in this section.

**Lemma 4.1** *The linear relation  $G$  is monotone if and only if*

$$(18) \quad \|y\|^2 - \|y^*\|^2 \geq 0, \text{ whenever}$$

$$(19) \quad (A + B)y + (B - A)y^* = 0.$$

*Consequently, if  $G$  is monotone then the  $p \times n$  matrix  $B - A$  must have full column rank, namely  $n$ .*

*Proof.* Define

$$P = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

It is easy to see that  $G$  is monotone if and only if

$$\langle (x, x^*), P \begin{pmatrix} x \\ x^* \end{pmatrix} \rangle \geq 0,$$

whenever  $Ax + Bx^* = 0$ . Define the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \text{Id} & -\text{Id} \\ \text{Id} & \text{Id} \end{pmatrix}$$

and put

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = Q \begin{pmatrix} y \\ y^* \end{pmatrix}.$$

Then  $G$  is monotone if and only if

$$(20) \quad \|y\|^2 - \|y^*\|^2 \geq 0, \text{ whenever}$$

$$(21) \quad (A + B)y + (B - A)y^* = 0.$$

If  $(B - A)$  does not have full column rank, then there exists  $y^* \neq 0$  such that  $(B - A)y^* = 0$ . Then  $(0, y^*)$  satisfies (21) but (20) fails. Therefore,  $B - A$  has to be full column rank. ■

**Theorem 4.2 (Minty parametrization)** *Assume that  $G$  is a monotone operator. Then  $(x, x^*) \in \text{gra } G$  if and only if*

$$(22) \quad x = \frac{1}{2}[\text{Id} + (B - A)^\dagger(B + A)]y$$

$$(23) \quad x^* = \frac{1}{2}[\text{Id} - (B - A)^\dagger(B + A)]y$$

*for  $y = x + x^* \in \text{ran}(\text{Id} + G)$ . Here the Moore-Penrose inverse  $(B - A)^\dagger = [(B - A)^\top(B - A)]^{-1}(B - A)^\top$ . In particular, when  $G$  is maximally monotone, we have*

$$\text{gra } G = \{((B - A)^{-1}By, -(B - A)^{-1}Ay) \mid y \in \mathbb{R}^n\}.$$

*Proof.* As  $(B - A)$  is full column rank,  $(B - A)^\top(B - A)$  is invertible. It follows from (19) that  $(B - A)^\top(A + B)y + (B - A)^\top(B - A)y^* = 0$  so that

$$y^* = -((B - A)^\top(B - A))^{-1}(B - A)^\top(A + B)y = -(B - A)^\dagger(A + B)y.$$

Then

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(y - y^*) = \frac{1}{\sqrt{2}}[\text{Id} + (B - A)^\dagger(B + A)]y \\ x^* &= \frac{1}{\sqrt{2}}(y + y^*) = \frac{1}{\sqrt{2}}[\text{Id} - (B - A)^\dagger(B + A)]y \end{aligned}$$

where  $y = \frac{x+x^*}{\sqrt{2}}$  with  $(x, x^*) \in \text{gra } G$ . Since  $\text{ran}(\text{Id} + G)$  is a subspace, we have

$$\begin{aligned} x &= \frac{1}{2}[\text{Id} + (B - A)^\dagger(B + A)]\tilde{y} \\ x^* &= \frac{1}{2}[\text{Id} - (B - A)^\dagger(B + A)]\tilde{y} \end{aligned}$$

with  $\tilde{y} = x + x^* \in \text{ran}(\text{Id} + G)$ .

If  $G$  is maximally monotone, then  $p = n$  by Theorem 3.6 and hence  $B - A$  is invertible, thus  $(B - A)^\dagger = (B - A)^{-1}$ . Moreover,  $\text{ran}(G + \text{Id}) = \mathbb{R}^n$ . Then (22) and (23) transpire to

$$(24) \quad x = \frac{1}{2}(B - A)^{-1}[B - A + (B + A)]y = (B - A)^{-1}By$$

$$(25) \quad x^* = \frac{1}{2}(B - A)^{-1}[(B - A) - (B + A)]y = -(B - A)^{-1}Ay$$

for  $y \in \mathbb{R}^n$ . ■

**Remark 4.3** See Lemma 2.7 for  $\text{ran}(G + \text{Id})$ . Note that as  $G$  is a monotone linear relation, the mapping

$$z \mapsto ((G + \text{Id})^{-1}, \text{Id} - (G + \text{Id})^{-1})(z)$$

is bijective and linear from  $\text{ran}(G + \text{Id})$  to  $\text{gra } G$ , therefore  $\dim(\text{ran}(G + \text{Id})) = \dim(\text{gra } G)$ .

**Corollary 4.4** *Let  $G$  be a monotone operator. Then  $\tilde{G}$  define in Corollary 3.4, the maximally monotone extension of  $G$ , has its Minty parametrization given by*

$$\text{gra } \tilde{G} = \{((V_g B - V_g A)^{-1}V_g B y, -(V_g B - V_g A)^{-1}V_g A y) \mid y \in \mathbb{R}^n\}$$

where  $V_g$  is given as in Corollary 3.4.

*Proof.* Since  $\text{rank}(V_g) = n$  and  $\text{rank}(A \ B) = p$ , by Lemma 2.3(5),  $\text{rank}(V_g A \ V_g B) = n$ . Apply Corollary 3.4 and Theorem 4.2 directly. ■



**Corollary 4.5** *When  $G$  is maximally monotone,*

$$\operatorname{dom} G = (B - A)^{-1}(\operatorname{ran} B), \quad \operatorname{ran} G = (B - A)^{-1}(\operatorname{ran} A).$$

Recall that  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \quad \forall x, y \in \operatorname{dom} T.$$

In terms of matrices

**Corollary 4.6** *Suppose that  $p = n$ ,  $AB^\top + BA^\top$  is negative semidefinite. Then  $(B - A)^{-1}B$  and  $-(B - A)^{-1}A$  are firmly nonexpansive.*

*Proof.* By Theorem 3.6,  $G$  is maximal monotone. Theorem 4.2 gives that

$$(B - A)^{-1}B = (\operatorname{Id} + G)^{-1}, \quad -(B - A)^{-1}A = (\operatorname{Id} + G^{-1})^{-1}.$$

Being resolvent of monotone operators  $G, G^{-1}$ , they are firmly nonexpansive, see [1, 13] or [4, Fact 2.5]. ■

## 5 Maximally monotone extensions with the same domain or the same range

The purpose of this section is to find maximal monotone linear subspace extensions of  $G$  which keep either  $\operatorname{dom} G$  or  $\operatorname{ran} G$  unchanged. For a closed convex set  $S \subseteq \mathbb{R}^n$ , let  $N_S$  denote its normal cone mapping.

**Proposition 5.1** *Assume that  $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a monotone linear relation. Then*

(i)  $T_1 = T + N_{\operatorname{dom} T}$ , *i.e.*,

$$x \mapsto T_1 x = \begin{cases} Tx + (\operatorname{dom} T)^\perp & \text{if } x \in \operatorname{dom} T \\ \emptyset & \text{otherwise} \end{cases}$$

*is maximally monotone. In particular,  $\operatorname{dom} T_1 = \operatorname{dom} T$ .*

(ii)  $T_2 = (T^{-1} + N_{\operatorname{ran} T})^{-1}$  *is a maximally monotone extension of  $T$  and  $\operatorname{ran} T_2 = \operatorname{ran} T$ .*

*Proof.* (i): Since  $0 \in T0 \subseteq (\text{dom } T)^\perp$  by [2, Proposition 2.2(i)], we have  $T_1 0 = T0 + (\text{dom } T)^\perp = (\text{dom } T)^\perp$  so that  $\text{dom } T_1 = \text{dom } T = (T_1 0)^\perp$ . Hence  $T_1$  is maximally monotone by Fact 2.5.

(ii): Apply (i) to  $T^{-1}$  to see that  $T^{-1} + N_{\text{ran } T}$  is a maximally monotone extension of  $T^{-1}$  with  $\text{dom}(T^{-1} + N_{\text{ran } T}) = \text{ran } T$ . Therefore,  $T_2$  is a maximally monotone extension of  $T$  with  $\text{ran } T_2 = \text{ran } T$ .  $\blacksquare$

Since

$$\text{gra } G = \{(x, x^*) \mid Ax + Bx^* = 0\}$$

we can use Gaussian elimination to reduce (A B) to row echelon form. Then back substitution to solve basic variables in terms of the free variables, see [14, page 61]. Row-echelon form gives

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = h_1 y_1 + \cdots + h_{2n-p} y_{2n-p} = \begin{pmatrix} C \\ D \end{pmatrix} y$$

where  $y \in \mathbb{R}^{2n-p}$  and

$$\begin{pmatrix} C \\ D \end{pmatrix} = (h_1, \dots, h_{2n-p})$$

with  $C, D$  being  $n \times (2n - p)$  matrices. Therefore,

$$(26) \quad \text{gra } G = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} \mid y \in \mathbb{R}^{2n-p} \right\}.$$

Define

$$(27) \quad \text{gra } E_1 = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} + \begin{pmatrix} 0 \\ (\text{ran } C)^\perp \end{pmatrix} \mid y \in \mathbb{R}^{2n-p} \right\}.$$

$$(28) \quad \text{gra } E_2 = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} + \begin{pmatrix} (\text{ran } D)^\perp \\ 0 \end{pmatrix} \mid y \in \mathbb{R}^{2n-p} \right\}.$$

**Theorem 5.2** (i)  $E_1$  is a maximally monotone extension of  $G$  with  $\text{dom } E_1 = \text{dom } G$ .  
Moreover,

$$(29) \quad \text{gra } E_1 = \text{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ (\text{ran } C)^\perp \end{pmatrix} = \text{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ \ker C^\top \end{pmatrix}.$$

(ii)  $E_2$  is a maximally monotone extension of  $G$  with  $\text{ran } E_2 = \text{ran } G$ . Moreover,

$$(30) \quad \text{gra } E_2 = \text{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} (\text{ran } D)^\perp \\ 0 \end{pmatrix} = \text{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} \ker D^\top \\ 0 \end{pmatrix}.$$

*Proof.* (i): Note that  $\text{dom } G = \text{ran } C$ . The maximal monotonicity follows from Proposition 5.1. (29) follows from (27) and that  $(\text{ran } C)^\perp = \ker C^\top$  [14, page 405].

(ii): Apply (i) to  $G^{-1}$ , i.e.,

$$(31) \quad \text{gra } G^{-1} = \left\{ \begin{pmatrix} Dy \\ Cy \end{pmatrix} \mid y \in \mathbb{R}^{2n-p} \right\}$$

and followed by taking the set-valued inverse. ■

Apparently, both extensions  $E_1, E_2$  rely on  $\text{gra } G, \text{dom } G, \text{ran } G$ , not on the  $A, B$ . In this sense,  $E_1, E_2$  are intrinsic maximal monotone linear subspace extensions.

**Remark 5.3** Theorem 5.2 is much easier to use than Corollary 3.8 when  $G$  is written in the form of (26). Indeed, it is not hard to check that

$$(32) \quad \text{gra}(E_1^*) = \{(B^\top u, -A^\top u) \mid B^\top u \in \text{dom } G, u \in \mathbb{R}^p\}.$$

$$(33) \quad \text{gra}(E_2^*) = \{B^\top u, -A^\top u \mid A^\top u \in \text{ran } G, u \in \mathbb{R}^p\}.$$

According to Fact 2.6,  $E_i^*$  is maximal monotone and  $\dim E_i^* = n$ . This implies that

$$\dim\{u \in \mathbb{R}^p \mid B^\top u \in \text{dom } G\} = n, \quad \dim\{u \in \mathbb{R}^p \mid A^\top u \in \text{ran } G\} = n.$$

Let  $M_i \in \mathbb{R}^{p \times p}$  with  $\text{rank } M = n$  and

$$(34) \quad \{u \in \mathbb{R}^p \mid B^\top u \in \text{dom } G\} = \text{ran } M_1,$$

$$(35) \quad \{u \in \mathbb{R}^p \mid A^\top u \in \text{ran } G\} = \text{ran } M_2.$$

Corollary 3.8 shows that

$$\text{gra } E_i = \{(x, x^*) \mid M_i^\top Ax + M_i^\top Bx^* = 0\}.$$

However, finding  $M_i$  from (34) and (35) may not be easy as it seems.

## 6 Examples

In the final section, we illustrate our maximally monotone extensions by considering three examples. In particular, they show that maximal monotone extensions  $\tilde{G}$  rely on the representation of  $G$  in terms of  $A, B$  and choices of  $N$  we shall use. However, the maximal monotone extensions  $E_i$  are intrinsic, only depending on  $\text{gra } G$ .

**Example 6.1** Consider

$$\text{gra } G = \left\{ \begin{pmatrix} x \\ x^* \end{pmatrix} \mid \begin{pmatrix} \text{Id} \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ C \end{pmatrix} x^* = 0, x, x^* \in \mathbb{R}^n \right\}$$

where  $C$  is a  $n \times n$  symmetric, positive definite matrix. Clearly,

$$\text{gra } G = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

We have

(i) For every  $\alpha \in [-1, 1]$ ,  $\tilde{G}_\alpha$  defined by

$$\text{gra } \tilde{G}_\alpha = \begin{cases} \{(0, \mathbb{R}^n)\}, & \text{if } \alpha = 1; \\ \{(x, \frac{1+\alpha}{1-\alpha}C^{-1}x) \mid x \in \mathbb{R}^n\}, & \text{otherwise} \end{cases}$$

is a maximally monotone linear extension of  $G$ .

(ii)  $E_1 = \tilde{G}_1$  and  $E_2 = \tilde{G}_{-1}$ .

*Proof.* (i): To find  $\tilde{G}_\alpha$ , we need eigenvectors of

$$\mathbf{A} = \begin{pmatrix} \text{Id} \\ 0 \end{pmatrix} (0 \ C^\top) + \begin{pmatrix} 0 \\ C \end{pmatrix} (\text{Id} \ 0) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$$

Counting multiplicity, the positive definite matrix  $C$  has eigen-pairs  $(\lambda_i, w_i)$  ( $i = 1, \dots, n$ ) such that  $\lambda_i > 0$ ,  $\|w_i\| = 1$  and  $\langle w_i, w_j \rangle = 0$  for  $i \neq j$ . As such, the matrix  $\mathbf{A}$  has  $2n$  eigen-pairs, namely

$$(\lambda_i, \begin{pmatrix} w_i \\ w_i \end{pmatrix})$$

and

$$(-\lambda_i, \begin{pmatrix} w_i \\ -w_i \end{pmatrix})$$

with  $i = 1, \dots, n$ . Put  $W = [w_1 \ \dots \ w_n]$  and write

$$V = \begin{pmatrix} W & W \\ W & -W \end{pmatrix}.$$

Then

$$W^\top C W = D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

In Theorem 3.7, take

$$N_\alpha = \begin{pmatrix} \mathbf{0} & \alpha \text{Id} \\ \mathbf{0} & \text{Id} \end{pmatrix}.$$

We have  $\text{rank } N_\alpha = n$ ,

$$N_\alpha^\top \text{Id}_\lambda N_\alpha = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\alpha^2 - 1)W^\top CW \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\alpha^2 - 1)D \end{pmatrix}$$

being negative semidefinite, and

$$VN_\alpha = \begin{pmatrix} 0 & (1 + \alpha)W \\ 0 & (\alpha - 1)W \end{pmatrix}.$$

Then by Theorem 3.7, we have an maximally monotone linear extension  $\tilde{G}_\alpha$  given by

$$\begin{aligned} \text{gra } \tilde{G}_\alpha &= \left\{ (x, x^*) \mid \begin{pmatrix} 0 \\ (1 + \alpha)W^\top x + (\alpha - 1)W^\top Cx^* \end{pmatrix} = 0 \right\} \\ &= \{(x, x^*) \mid (1 + \alpha)x + (\alpha - 1)Cx^* = 0\} \\ &= \begin{cases} \{(0, \mathbb{R}^n)\}, & \text{if } \alpha = 1; \\ \{(x, \frac{1+\alpha}{1-\alpha}C^{-1}x) \mid x \in \mathbb{R}^n\}, & \text{otherwise} \end{cases} \end{aligned}$$

Hence we get the result as desire.

(ii): It is immediate from Theorem 5.2 and (i). ■

**Example 6.2** Consider

$$\text{gra } G = \left\{ \begin{pmatrix} x \\ x^* \end{pmatrix} \mid \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 0, x_i, x_i^* \in \mathbb{R} \right\}.$$

Then

(i)

$$\tilde{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1+\sqrt{2}}{2-\sqrt{2}} \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{2}}{10} \end{pmatrix}$$

are the maximally monotone extensions of  $G$ .

(ii)

$$E_1(x_1, 0) = (x_1, \mathbb{R}) \quad \forall x_1 \in \mathbb{R}.$$

(iii)

$$E_2(x_1, y) = (x_1, 0) \quad \forall x_1, y \in \mathbb{R}.$$

*Proof.* We have

$$\text{gra } G = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\}$$

is monotone. Since  $\dim G = 1$ ,  $G$  is not maximally monotone by Fact 2.5.

The matrix

$$AB^\top + BA^\top = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

has positive eigenvalue  $-1 + \sqrt{2}$  with eigenvector

$$u = \begin{pmatrix} 0 \\ 1 \\ 1 - \sqrt{2} \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} B^\top \\ A^\top \end{pmatrix} u = \begin{pmatrix} 0 \\ 2 - \sqrt{2} \\ 0 \\ -1 + \sqrt{2} \end{pmatrix}.$$

Then by Corollary 3.4,  $\text{gra } \tilde{G}_1 =$

$$\left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} + \left\{ \begin{pmatrix} 0 \\ 2 - \sqrt{2} \\ 0 \\ -1 + \sqrt{2} \end{pmatrix} x_2 \mid x_2 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x_1 \\ (2 - \sqrt{2})x_2 \\ x_1 \\ (-1 + \sqrt{2})x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}.$$

Therefore,

$$\tilde{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1 + \sqrt{2}}{2 - \sqrt{2}} \end{pmatrix}.$$

We have

$$(36) \quad \text{Id}_\lambda = \begin{pmatrix} -1 + \sqrt{2} & 0 & 0 \\ 0 & -1 - \sqrt{2} & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{-1 + \sqrt{2}} & -\frac{1}{-1 - \sqrt{2}} & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Take

$$(37) \quad N = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We have  $\text{rank } N = 2$  and

$$(38) \quad N^\top \text{Id}_\lambda N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -7 - 3\sqrt{2} & 1 + \sqrt{2} \\ 0 & 1 + \sqrt{2} & -4 \end{pmatrix},$$

being negative semidefinite.

By Theorem 3.7, with  $V, N$  given in (36) and (37), we use the NullSpace command in Maple to solve

$$(VN)^\top Ax + (VN)^\top Bx^* = 0,$$

and get

$$\text{gra } \tilde{G}_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2\sqrt{2} \\ 5\sqrt{2} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\text{Thus } \tilde{G}_2 = \begin{pmatrix} 1 & -2\sqrt{2} \\ 0 & 5\sqrt{2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{2}}{10} \end{pmatrix}.$$

On the other hand,

$$\text{gra } E_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbb{R} \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ \mathbb{R} \end{pmatrix} \mid x_1 \in \mathbb{R} \right\}$$

gives

$$E_1(x_1, 0) = (x_1, \mathbb{R}) \quad \forall x_1 \in \mathbb{R}.$$

And

$$\text{gra } E_2 = \left\{ \begin{pmatrix} x_1 \\ \mathbb{R} \\ x_1 \\ 0 \end{pmatrix} \mid x_1 \in \mathbb{R} \right\}.$$

gives

$$E_2(x_1, y) = (x_1, 0) \quad \forall x_1, y \in \mathbb{R}.$$

■

In [5], the authors use autoconjugates to find maximally monotone extensions of monotone operators. In general, it is not clear whether the maximally monotone extensions of a linear relation is still a linear relation. As both monotone operators in Examples 6.2 and

Examples 6.1 are subset of  $\{(x, x) \mid x \in \mathbb{R}^n\}$ , [5, Example 5.10] shows that the maximally monotone extension obtained by autoconjugate must be Id, which are different from the ones given here.

**Example 6.3** Set  $\text{gra } G = \{(x, x^*) \mid Ax + Bx^* = 0\}$  where

$$(39) \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 \\ 1 & 7 \\ 0 & 2 \end{pmatrix}, \text{ thus } (A \ B) = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 0 & 1 & 7 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

Then

$$\tilde{G}_1 = \begin{pmatrix} \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} & \frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} \frac{33}{4} - \frac{\sqrt{201}}{6} & \frac{13}{4} - \frac{\sqrt{201}}{6} \\ -\frac{29}{20} + \frac{\sqrt{201}}{30} & -\frac{9}{20} + \frac{\sqrt{201}}{30} \end{pmatrix}.$$

are two maximally monotone linear extensions of  $G$ .

Moreover,

$$\text{gra } E_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} x_2 \mid x_1, x_2 \in \mathbb{R} \right\}, \quad \text{gra } E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} x_2 \mid x_1, x_2 \in \mathbb{R} \right\}.$$

*Proof.* We have  $\text{rank}(A \ B) = 3$  and

$$(40) \quad \text{Id}_\lambda = \begin{pmatrix} 13 + \sqrt{201} & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 13 - \sqrt{201} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{20}{1+\sqrt{201}} & 0 & \frac{20}{1-\sqrt{201}} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$(41) \quad V_g = \begin{pmatrix} 0 & -1 & 1 \\ \frac{20}{1-\sqrt{201}} & 1 & 1 \end{pmatrix}.$$

Clearly, here  $p = 3, n = 2$  and  $AB^\top + BA^\top$  has exactly  $p - n = 3 - 2 = 1$  positive eigenvalue. By Theorem 3.3(i)(v),  $G$  is monotone.

Since  $AB^\top + BA^\top$  is not negative semidefinite, by Theorem 3.6(i)(iii),  $G$  is not maximally monotone. With  $V_g$  given in (41) and  $A, B$  in (39), use the NullSpace command in maple to solve  $V_g Ax + V_g Bx^* = 0$  and obtain  $\tilde{G}_1$  defined by

$$\text{gra } \tilde{G}_1 = \text{span} \left\{ \begin{pmatrix} -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} \\ 0 \\ 1 \end{pmatrix} \right\}.$$



By Corollary 3.4,  $\tilde{G}_1$  is a maximally monotone linear subspace extension of  $G$ . Then

$$\tilde{G}_1 = \begin{pmatrix} -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} & \frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}.$$

Let  $N$  be defined by

$$(42) \quad N = \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then  $\text{rank } N = 2$  and

$$N^\top \text{Id}_\lambda N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & \frac{338-24\sqrt{201}}{25} \end{pmatrix}.$$

is negative semidefinite.

With  $N$  in (42),  $A, B$  in (39) and  $V$  in (40), use the NullSpace command in maple to solve  $(VN)^\top Ax + (VN)^\top Bx^* = 0$ . By Theorem 3.7, we get a maximally monotone linear extension of  $G$ ,  $\tilde{G}_2$ , defined by

$$\tilde{G}_2 = \begin{pmatrix} -\frac{9}{20} + \frac{\sqrt{201}}{30} & -\frac{13}{4} + \frac{\sqrt{201}}{6} \\ \frac{29}{20} - \frac{\sqrt{201}}{30} & \frac{33}{4} - \frac{\sqrt{201}}{6} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{33}{4} - \frac{\sqrt{201}}{6} & \frac{13}{4} - \frac{\sqrt{201}}{6} \\ -\frac{29}{20} + \frac{\sqrt{201}}{30} & -\frac{9}{20} + \frac{\sqrt{201}}{30} \end{pmatrix}.$$

To find  $E_1$  and  $E_2$ , using the LinearSolve command in Maple, we get  $\text{gra } G = \text{ran} \begin{pmatrix} C \\ D \end{pmatrix}$ , where

$$C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad D = \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

It follows from Theorem 5.2 that

$$\begin{aligned} \text{gra } E_1 &= \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} x_2 \mid x_1, x_2 \in \mathbb{R} \right\}, \\ \text{gra } E_2 &= \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} x_2 \mid x_1, x_2 \in \mathbb{R} \right\}. \end{aligned}$$

■

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